AIAA 82-4247

An Analytic Approach to Two-Fixed-Impulse Transfers Between Keplerian Orbits

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Solutions are obtained for the two-impulse transfer of a vehicle between arbitrary inclined orbits in an inverse-square force field with the restriction that the magnitude of each of the two impulses has a fixed preassigned value. The two magnitudes need not be equal. The equations for the conservation of angular momentum and energy are augmented by the Laplace integral; these equations establish linear relationships between several of the variables. This set of linear equations and one of two quadratic equations constitute the analytically tractable part of the solution. The remaining part, consisting of finding the zeros (if they exist) of a single trigonometric function of one variable, is solved using numerical methods. Explicit lower bounds on each of the magnitudes of the two impulses are obtained by requiring the solution be real. Graphical results are presented to illustrate the solution.

I. Introduction

THE problem consists of transferring a vehicle between two noncoplanar orbits in an inverse-square force field with a propulsion system consisting of two fixed, preassigned impulses. The time of flight is not specified. The problem arises by assuming the simplest form of propulsion system necessary to achieve an orbital transfer. Fixing the magnitudes of the two impulses allows the propulsion system to be classified by specific impulse values. These characteristics are highly desirable with respect to targeting, propulsion, and guidance of an interim upper-stage vehicle proposed for transferring payloads in space.

A previous analysis of this problem was given by Grubin.[1] His initial approach using only the angular momentum and energy integrals seemingly removed the intractable portion of the problem and reduced it to one that requires only the solution of a quartic equation. He chose, however, not to analytically solve this quartic, but to numerically iterate for a solution. Lee[2] and Escobal[3] indicated that an additional constraint equation is required for a physically meaningful problem. Grubin,[1] recognizing the need for releasing a parameter on the initial or final orbit, chose the eccentric anomaly on the final orbit. He numerically performed a double iteration to obtain a solution.

In this analysis, an additional constraint equation is obtained from the Laplace integral. This constraint allows the tractable portion of the problem to be reduced to the solution of a single quadratic equation. The remaining portion of the problem is to determine the real zeros (if they exist) of a single, trigonometric function of the true anomaly in either the initial or final orbit. This trigonometric function is scanned to isolate its zeros, and then a Newton-Raphson technique is used to determine them to the desired accuracy. Lower bounds on the magnitudes of the two impulses are obtained explicitly by requiring real solutions to the problem.

For a fixed position in the initial orbit and for sufficiently large impulsive magnitudes, a given problem appears to have, at most, eight possible solutions; four from each sign of the radical, resulting from the solution of the quadratic equation. The technique is illustrated by numerical examples; graphs are included to enhance the exposition.

Another solution to this problem with practical examples for conceptual mission design was developed by Chu et al.[4] Assuming different constraints and operational requirements, sixteen discrete solutions were determined for a given mission.

II. Definition of the Problem

We are concerned with the following problem. Is there a two-impulse transfer trajectory in an inverse-square gravitational field from a given position vector (R_1) on a given initial orbit (O_i) to a given position vector (R_2) on a given final orbit (O_f), assuming that the two impulses have fixed magnitudes C_1 and C_2, respectively? It will be seen that the answer to this question is, in general, no. If a condition of the problem is relaxed by not requiring that the final position vector R_2 be fixed but be free to be any point on the orbit O_f, then, as will be seen, there may be solutions. In fact, there may be as many as eight solutions. The mathematical formulation of the problem follows.

Let V_1^- be the velocity vector at an arbitrary point on O_i and V_2^+ the velocity vector at an arbitrary point on O_f. V_1^+ will be the velocity vector on the transfer orbit (O_t) at the instant immediately following the firing of the first impulse; V_2^- will be the velocity vector on O_t at the instant immediately preceding the firing of the second impulse. The difference

$$\Delta V_1 = V_1^+ - V_1^- \tag{1a}$$

is the change in the velocity vector due to firing the first impulse. The magnitude of this impulse is given as

$$|\Delta V_1| = C_1 \tag{2a}$$

Similarly, the difference

$$\Delta V_2 = V_2^+ - V_2^- \tag{1b}$$

is the change in the velocity vector due to firing the second impulse. The magnitude of this impulse is given as

$$|\Delta V_2| = C_2 \tag{2b}$$

Received June 29, 1981; revision received February 16, 1982.

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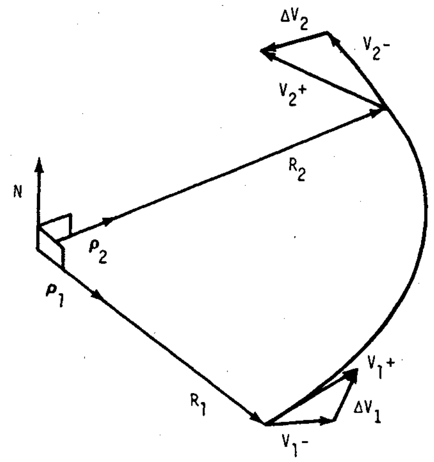

Fig. 1 Impulsive geometry.

We define a coordinate system using the plane of the transfer orbit (Fig. 1). Let ρ_1 be a unit vector in the direction of the initial position vector R_1, ρ_2 a unit vector in the direction of the final position vector R_2, and N a unit vector normal to the plane defined by ρ_1 and ρ_2. It is assumed throughout that ρ_1 and ρ_2 are not collinear. This triad of unit vectors is

$$\rho_1=\frac{R_1}{r_1},\qquad \rho_2=\frac{R_2}{r_2},\qquad N=\frac{\rho_1\times\rho_2}{s} \tag{3}$$

where $r=|R|$ and $s=|\rho_1\times\rho_2|$. Note that the transfer orbit coordinate system is, in general, not an orthogonal coordinate system. The impulse vectors [Eq. (1)], expressed in this coordinate system have the form

$$\Delta V_1=x_1\rho_1+y_1\rho_2+z_1N \qquad \Delta V_2=x_2\rho_1+y_2\rho_2+z_2N \tag{4}$$

where x_i, y_i, and $z_i\,(i=1,2)$, the components of the impulse vectors in the directions of the unit vectors ρ_1, ρ_2, and N, respectively, are to be determined.

The problem as stated cannot be solved except for special boundary conditions. For a two-impulse transfer problem in an inverse-square force field, it has been shown by Lee[2] and Escobal[3] that the magnitude of the two impulses can be expressed as

$$|\Delta V_1|=U_1(p),\qquad |\Delta V_2|=U_2(p)$$

where $U_1(p)$ and $U_2(p)$ are two known functions, and p is the parameter of the transfer orbit. Since there is only one free variable p, the magnitudes of the two impulses cannot both be specified arbitrarily. A constraint on either the initial or final orbit must be removed to achieve a solution with the constraints specified by Eq. (2). We shall arbitrarily choose the true anomaly θ on the final orbit as this free variable. Other choices are certainly possible.

Two questions now arise. What variables are required to define a solution to this problem? What are the equations of constraint? From Eq. (4) we recognize six unknowns (x_1, y_1, z_1, x_2, y_2, and z_2); and with the final, true anomaly θ the problem has seven variables. We need to define seven independent equations. The constraint on the magnitudes of the two impulses [Eq. (2)] represents two equations. The additional five equations are obtained from two first integrals of motion for an inverse-square force field and the Laplace integral.

III. Necessary Equations for a Solution

The development in the first two parts of this section, concerned with using the angular momentum and energy integrals, is due to Grubin.[1] It is included to make this presentation self-contained.

A. Angular Momentum Integral

From the conservation of angular momentum, we have the vector relationship

$$R_1\times V_1^+=R_2\times V_2^- \tag{5}$$

Substituting for V_1^+ and V_2^- from Eq. (1), we obtain

$$R_1\times(V_1^-+\Delta V_1)=R_2\times(V_2^+-\Delta V_2)$$

But $R_1\times V_1^-=H_i$ and $R_2\times V_2^+=H_f$ where H_i and H_f are the angular momentum vectors on the initial and final orbits, respectively. These vectors are evaluated from the boundary conditions and are constants of the solution. From Eq. (4), we substitute for the vectors ΔV_1 and ΔV_2 and obtain a vector equation in terms of the unknown components of the impulse vector.

$$R_1\times(x_1\rho_1+y_1\rho_2+z_1N)+R_2\times(x_2\rho_1+y_2\rho_2+z_2N)=\Delta H$$

where $\Delta H=H_f-H_i$. Using Eq. (3), this equation may be simply expressed as

$$r_1[sy_1N+z_1(\rho_1\times N)]+r_2[-sx_2N+z_2(\rho_2\times N)]=\Delta H \tag{6}$$

The three independent scalar equations embedded in Eq. (6) are obtained by taking the scalar product of ρ_1, ρ_2, and N, respectively, with Eq. (6). Dotting first by ρ_1 and ρ_2, we obtain, respectively,

$$(\rho_1\times\rho_2)\cdot(z_2N-V_2^+)=0 \qquad (\rho_2\times\rho_1)\cdot(z_1N+V_1^-)=0$$

Since the vectors ρ_1 and ρ_2 are not collinear, the general solution of these equations is

$$z_1=-N\cdot V_1^- \qquad z_2=N\cdot V_2^+ \tag{7}$$

These equations are physically clear. They indicate that the out-of-plane components z_1 and z_2 of the impulse vectors ΔV_1 and ΔV_2, respectively, must nullify the out-of-plane components of the initial and final velocity vectors V_1^- and V_2^+, respectively. Note that the solution for the components z_1 and z_2 are independent of the remaining components of the impulse vectors; i.e., (x_1, y_1, x_2, and y_2).

Taking the scalar product of the vector N with Eq. (6), we obtain the third independent relationship

$$r_1y_1-r_2x_2=b_1 \tag{8}$$

where $b_1=N\cdot\Delta H/s$.

B. Energy Integral

From the conservation of energy, we have the scalar relationship

$$\tfrac{1}{2}V_1^+\cdot V_1^+-\mu/r_1=\tfrac{1}{2}V_2^-\cdot V_2^--\mu/r_2$$

where μ is the gravitational constant.

Proceeding in a manner similar to that in Sec. IIIA, we obtain the single constraint equation

$$a_1x_1+a_2y_1+a_3x_2+a_4y_2=b_2 \tag{9}$$

where

$$a_1=\rho_1\cdot V_1^-,\quad a_2=\rho_2\cdot V_1^-,\quad a_3=\rho_1\cdot V_2^+,\quad a_4=\rho_2\cdot V_2^+$$

$$b_2=E_f-E_i+C\tfrac{1}{2}(C_2^2-C_1^2)-(z_2^2-z_1^2)$$

and E_f and E_i, the Keplerian energies on the final and initial orbits, respectively, are given as

$$E_f = \tfrac{1}{2}(V_2^+ \cdot V_2^+) - (\mu/r_2) \qquad E_i = \tfrac{1}{2}(V_1^- \cdot V_1^-) - (\mu/r_1)$$

C. Laplace Integral

From the Laplace integral, we have the vector equation

$$-\mu\rho_1 + V_1^+ \times (R_1 \times V_1^+) = -\mu\rho_2 + V_2^- \times (R_2 \times V_2^-) \tag{10}$$

This equation represents a vector in the direction of pericenter of the transfer orbit whose magnitude is equal to μ times the eccentricity of the orbit. The two vector relationships, Eqs. (5) and (10), are not totally independent since the angular momentum vector is orthogonal to the Laplace vector. Using Eq. (5) in Eq. (10), we may write

$$(V_1^+ - V_2^-) \times (R_1 \times V_1^+) = -\mu(\rho_2 - \rho_1)$$

Taking the scalar product of the vector $(V_1^+ - V_2^-)$ (a vector in the transfer plane) with this equation, we obtain[1]

$$(V_1^+ - V_2^-) \cdot (\rho_1 - \rho_2) = 0 \tag{11}$$

A form of this equation can be found in Battin[5] (p. 109).

Using Eqs. (1) and (4) in this equation, we obtain a fifth linear relationship between the components x_1, y_1, x_2, and y_2

$$x_1 - y_1 + x_2 - y_2 = b_3 \tag{12}$$

where

$$b_3 = (a_2 - a_1 + a_3 - a_4)/(1-c)$$

and

$$c = \rho_1 \cdot \rho_2 = \cos(\rho_1, \rho_2) \tag{13}$$

D. Impulse Magnitude Constraints

The necessary sixth and seventh relationships between the unknown parameters are obtained by using Eqs. (1) and (4) in Eq. (2) to obtain

$$x_1^2 + y_1^2 + 2x_1y_1c = k_1^2 \tag{14a}$$

$$x_2^2 + y_2^2 + 2x_2y_2c = k_2^2 \tag{14b}$$

where

$$k_1^2 = C_1^2 - z_1^2 \qquad k_2^2 = C_2^2 - z_2^2$$

Equations (7), (8), (9), (12), and (14) are selected as the seven scalar constraint equations to determine a solution for the variables x_1, y_1, z_1, x_2, y_2, z_2, and θ.

Since θ, the true anomaly on the final orbit, occurs nonlinearly in the system of constraint equations, consider for the present that it is not a free parameter but remains fixed. We will first obtain a solution for the unknowns $(x_1, y_1, \ldots, z_2)$. In Grubin,[1] since Eq. (12) is absent, this involved solving a quartic equation. We will show that using the developed relationship expressed by Eq. (12), the solution for the unknown parameters $(x_1, y_1, \ldots, z_2)$ involves only a quadratic equation.

IV. Problem Solution

A. Analytic Solution

Equations (8), (9), (12), and (14b) comprise four independent equations in the four unknowns x_1, y_1, x_2, and y_2. The out-of-plane components, z_1 and z_2, are obtained from Eq. (7). Eliminating x_1, y_1, and x_2, the unknown y_2 is given by

$$y_2 = (-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha \tag{15a}$$

where

$$\alpha = q_1^2 + q_2^2 - 2q_1q_2c \qquad \beta = 2(cq_1 - q_2)q_3 \qquad \gamma = q_3^2 - k_2^2q_1^2$$

and the q's are defined as

$$q_1 = r_2(a_1 + a_2) + r_1(a_3 - a_1) \qquad q_2 = r_1(a_1 + a_4)$$

$$q_3 = r_1(b_2 - a_1b_3) - b_1(a_1 + a_2)$$

The variables x_2, x_1, and y_1 are

$$x_2 = \{r_1[b_2 - a_4y_2 - a_1(y_2 + b_3)] - b_1(a_1 + a_2)\}/q_1$$

$$x_1 = [b_2 + a_2b_3 + (a_2 - a_4)y_2 - (a_2 + a_3)x_2]/(a_1 + a_2) \tag{15b}$$

$$y_1 = [b_2 - a_1b_3 - (a_1 + a_4)y_2 + (a_1 - a_3)x_2]/(a_1 + a_2)$$

The quantity q_1 is the determinant of the coefficients of x_1, y_1, and x_2 in Eqs. (8), (9), and (12). No general physical interpretation has been deduced if q_1 is zero. For a Hohmann-type transfer [those for which the transfer angle is π and which is excluded from this analysis, see Eq. (3)] q_1 is equal to zero. It should be noted from Eq. (15a) that: 1) for a given position vector on the final orbit, there are two possible values for y_2 and, thus two possible solutions to the problem; and 2) for a real solution to the problem, the argument under the radical must be non-negative. This, as shown in the Appendix, leads to lower bounds for the constants C_1 and C_2.

The solution described by Eqs. (7) and (15) may not be physically realizable because we do not know if the values found satisfy Eq. (14a). After obtaining numerical values from his quartic, Grubin[1] discovered that the argument of pericenter ω on the transfer orbit took on different values when the state vectors (R_1, V_1^+) and (R_2, V_2^-) were used. This occurred because, as mentioned previously, the magnitude of the impulses C_1 and C_2, respectively, cannot be both constrained to given values without releasing some element on either the initial or final orbit. In the development given earlier, Eq. (10) constrains the argument of pericenter ω on the transfer orbit to be the same when the state vectors (R_1, V_1^+) and (R_2, V_2^-) are specified. We shall now develop physically realizable solutions; that is, show that Eq. (14a) can be satisfied by properly choosing the true anomaly θ on the final orbit.

B. Numerical Solution

We rewrite Eq. (14a) in the form

$$g(\theta) = \tfrac{1}{2}(x_1^2 + y_1^2 + 2x_1y_1c - k_1^2) \tag{16}$$

It is required to determine the values of θ, consistent with Eqs. (7) and (15), for which $g(\theta) = 0$. From the preceding analysis and the results of Lee[2] and Escobal,[3] the function $g(\theta)$ is known to be a trigonometric function of θ, but impractical to solve analytically. We proceed to obtain the zeros of Eq. (16) using a numerical method.

Since $g(\theta)$ is a periodic function, the final orbit is numerically scanned for the domain of θ, $0 \le \theta < 2\pi$. If a change in sign of the function is encountered, an approximate value of θ (obtained through linear interpolation) is stored. After the entire final orbit is scanned, a more precise numerical value for $g(\theta) = 0$ for each stored value of θ is obtained by a Newton-Raphson technique.

Expand the function $g(\theta)$ about one of the reference-stored values to obtain a differential change in the function at the reference value.

$$g'(\theta) = (x_1 + cy_1)x_1' + (y_1 + cx_1)y_1' + z_1 z_1' + x_1 y_1 c' \tag{17}$$

where the prime refers to differentiation with respect to θ. This derivative could be developed numerically with care in defining a proper numerical perturbation. However, if discontinuities occur in the function, numerical difficulties may be encountered. The derivatives c' and z_1' can be immediately obtained as follows. From Eq. (13), the derivative of c is

$$c' = \rho_1 \cdot \rho_2'$$

since only ρ_2 is a function of θ. However, from Battin[5] (p. 21), a unit vector in the direction of the position vector is

$$\rho_2 = \cos\theta\ \xi + \sin\theta\ \eta$$

where ξ and η are constant unit vectors on the final orbit. The derivative of ρ_2 with respect to θ is

$$\rho_2' = -\sin\theta\ \xi + \cos\theta\ \eta$$

From Eq. (7), the derivative of z_1 is

$$z_1' = -N' \cdot V_1^-$$

where

$$N' = (\rho_1 \times \rho_2' - s'N)/s$$

and

$$s' = -cc'/s$$

The other two derivatives, x_1' and y_1' [required for an evaluation of $g'(\theta)$] are obtained from a solution of the differential changes of Eqs. (8), (9), (12), and (14b). Expressed in vector-matrix form, this relationship is described as

$$M \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{bmatrix} = F \tag{18}$$

where

$$M = \begin{bmatrix} 0 & r_1 & -r_2 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & x_2 + cy_2 & y_2 + cx_2 \end{bmatrix}$$

$$F = \begin{bmatrix} x_2 r_2' + (\Delta H \cdot N' - b_1 s')/s \\ 2(z_1 z_1' - z_2 z_2') - y_1 a_2' - x_2 a_3' - y_2 a_4' \\ (a_2' + a_3' - a_4' + b_3 c')/(1-c) \\ -(x_2 y_2 c' + z_2 z_2') \end{bmatrix}$$

$$r_2' = (r_2^2 e_f \sin\theta)/p_f$$

and e_f and p_f are the eccentricity and parameter on the final orbit. The remaining derivatives are

$$z_2' = N \cdot V_2^+ \qquad a_2' = \rho_2' \cdot V_1^-$$

$$a_3' = \rho_1 \cdot (V_2^+)' \qquad a_4' = \rho_2 \cdot (V_2^+)' + V_2^+ \cdot \rho_2'$$

where, from Battin,[5] the velocity vector on the final orbit may be expressed as

$$V_2^+ = -(|H_f|/p_f)[\sin\theta\ \xi - (e_f + \cos\theta)\eta]$$

and its derivative with respect to θ as

$$(V_2^+)' = -(\mu\rho_2/|H_f|)$$

The derivatives x_1' and y_1' are obtained from the solution of Eq. (18) as

$$\begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{bmatrix} = M^{-1}F \tag{19}$$

Returning to the solution of the equation $g(\theta) = 0$ [Eq. (16)], using the derivative $g'(\theta)$ [Eq. (17)], we may write

$$g(\theta) - g(\theta_0) = g'(\theta_0)(\theta - \theta_0) \tag{20}$$

where θ_0 is a stored reference value. Since we have an approximate solution for θ (namely, θ_0) from the numerical scan of the function $g(\theta)$, the Newton-Raphson technique [as expressed in Eq. (20)] should converge in a few iterations. Successive values of θ are obtained from

$$\theta_{i+1} = \theta_i - \frac{g(\theta_i)}{g'(\theta_i)}, \quad i = 0,1,2,... \tag{21}$$

This completes the solution for the two-impulse transfer between two specified but arbitrary orbits with constraints on the magnitudes of the two impulses.

V. Illustrative Examples

The examples serve a threefold purpose. 1) They corroborate the solution obtained when the optimal ΔV magnitudes are used[6] as the constraining values for C_1 and C_2. 2) They indicate a boundary condition that should be avoided. 3) They illustrate the ease in obtaining solutions for arbitrary values of C_1 and C_2.

The problems consist of a noncoplanar transfer from a low elliptical orbit to a geosynchronous orbit. Complete information on the two orbits is given in Table 1. If we had chosen the line of nodes as the departure location from the initial orbit, the solution would fail. The reason is the N vector [Eq. (3)] remains fixed in direction when the initial position vector lies on the line of nodes. This geometry will produce solutions with all the plane changes either at the initial position or at the final position [Eq. (7)]. An initial elliptical orbit with a pericenter vector not on the line of nodes will avoid this difficulty. The optimal transfer orbit elements are also listed in Table 1. The magnitudes of the optimal impulse vectors are computed as[6]

$$|\Delta V_1| = 7934.29 \text{ fps}, \qquad |\Delta V_2| = 5816.97 \text{ fps}$$

We shall first use these ΔV magnitudes as the constraining values, C_1 and C_2, respectively. The function $g(\theta)$ for this problem is graphically illustrated in Fig. 2. Note that the function has two branches, 1) the left branch comes from

Table 1 Transfer to geosynchronous orbit

Orbit	Altitude of periapsis h_p, n.mi.	Altitude of apoapsis h_a, n.mi.	Inclination i, deg	Node Ω, deg	Argument of periapsis ω, deg	Departure true anomaly θ, deg
Initial	150	300	28.5	0	90	267.87
Final	19325.5	19325.5	0	–	–	–
Optimal transfer	224.86	19325.7	26.26	0.188	0.142	357.56

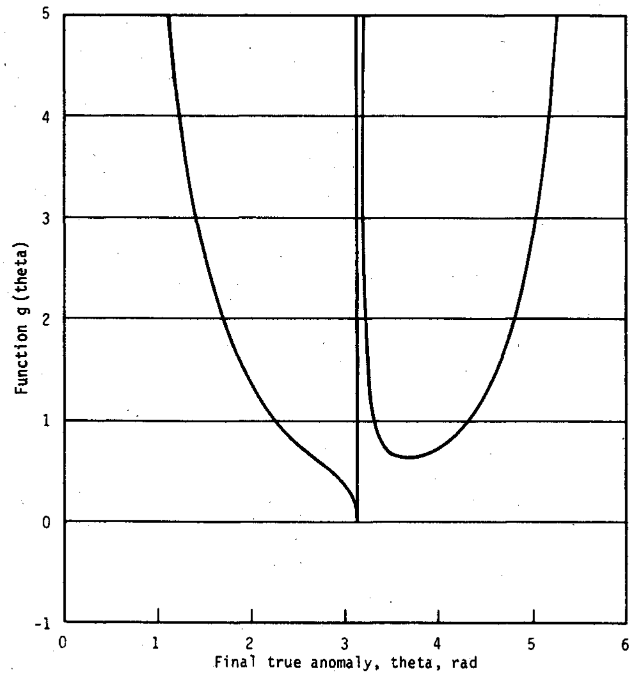

Fig. 2 Geosynchronous transfer $C_I = 7934.29$ fps; $C_2 = 5816.97$ fps.

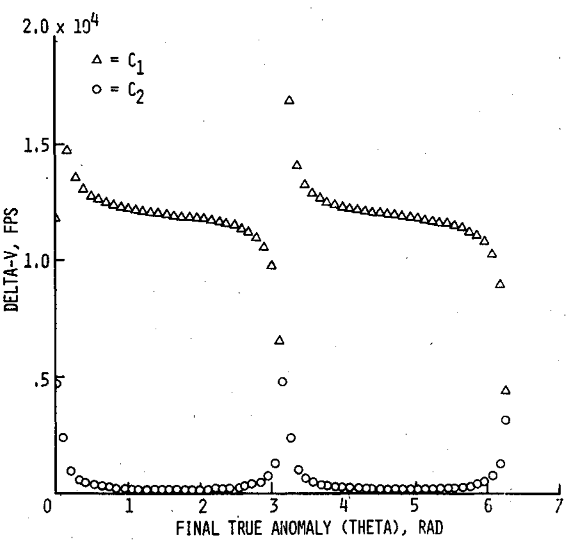

Fig. 3 Lower bounds on C_I and C_2.

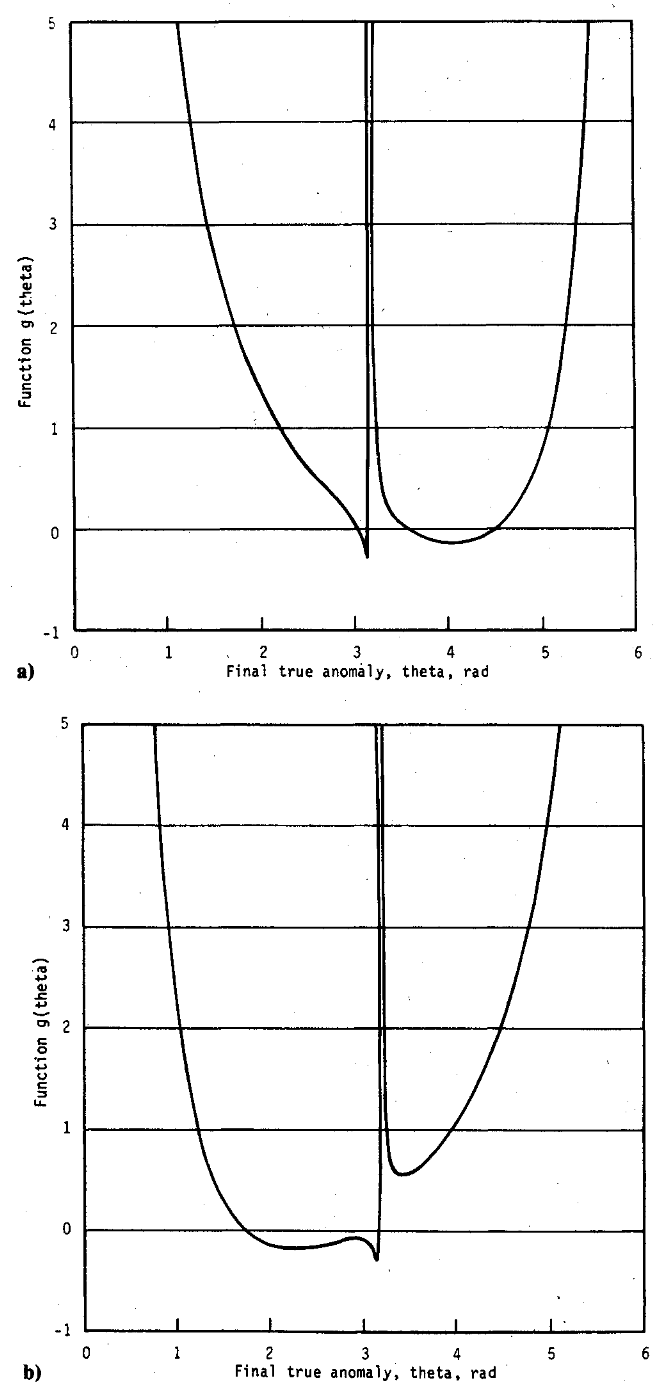

Fig. 4 Geosynchronous transfer $C_I = 20{,}000$ fps; $C_2 = 15{,}000$ fps; a) ISIGN = +1, b) ISIGN = −1.

infinity, crosses the θ axis at approximately $\theta = \pi$ and goes to infinity almost vertically, and 2) the right branch comes from and recedes to infinity without crossing the θ axis. The function $g(\theta)$ approaches infinity as the initial and final position vectors become nearly collinear. For the values for C_I and C_2, given earlier, the discriminant of the quadratic equation is zero, and there is only one solution for y_2.

The lower bounds on the magnitudes C_I and C_2 [computed via Eq. (A15)], for the range $0 \leq \theta < 2\pi$, are graphically illustrated in Fig. 3. The bounds for this problem result from the plane change requirements only. Note in Fig. 3 that for the values of C_I and C_2 specified, only a narrow window (located at $\theta \simeq 3$ rad) is available for a solution. By examining information such as, for example, Figs. 2 and 3, one can rapidly deduce 1) if solutions exist, and 2) the necessary minimum values for the magnitudes of C_1 and C_2 of these solutions.

As a second illustration we increase the values of C_1 and C_2 to

$$C_1 = 20{,}000 \text{ fps}, \qquad C_2 = 15{,}000 \text{ fps}$$

In Figs. 4a and 4b the function $g(\theta)$ is plotted for the plus and minus signs, respectively, of the radical in the solution of the quadratic equation. Note that there are six distinct solutions. These six solutions are characterized by 1) both elliptical and hyperbolic transfer orbits, 2) large plane changes, and 3) large energy transfer orbits. The second solution in each figure (proceeding from the left) is of further interest because it appears there may be a cusp in the function $g(\theta)$. This would certainly cause difficulty in the Newton-Raphson convergence technique. If this region is magnified, the function $g(\theta)$ is found to be smooth.

Conclusions

An analytic approach was used to develop solutions to the two-impulse transfer between Keplerian orbits with the restriction that the magnitudes of the two impulses are constrained to fixed values. From the Laplace integral, an additional linear relationship was developed that allows the tractable portion of the problem to be reduced to a solution of a quadratic equation. The remaining portion of the problem—finding the roots of a single, trigonometric function—is solved by a numerical method. Lower bounds are explicitly developed for the magnitudes of the two impulses. Graphical results are presented to 1) illustrate the method, and 2) indicate the ease in obtaining solutions.

Appendix: Lower Bounds on Impulse Magnitudes C_1 and C_2

Lower bounds on the magnitudes of the two impulses, C_1 and C_2, can be obtained by requiring that the solution to the two-impulse transfer problem be real.

From Eqs. (8), (9), (12), and (14b) of the text, we can obtain the following two equations in terms of the two unknowns, x_2 and y_2, as

$$\begin{aligned} q_1x_2 + q_2y_2 &= q_3 + (r_1/2)(C_2^2 - C_1^2) \\ x_2^2 + y_2^2 + 2cx_2y_2 &= C_2^2 - z_2^2 \end{aligned} \tag{A1}$$

By letting

$$d_1 = \frac{2q_1}{r_1}, \qquad d_2 = \frac{2q_2}{r_1}, \qquad d_3 = \frac{2q_3}{r_1}$$

Eqs. (A1) become

$$\begin{aligned} d_1x_2 + d_2y_2 &= d_3 + C_2^2 - C_1^2 \\ x_2^2 + y_2^2 + 2cx_2y_2 + z_2^2 &= C_2^2 \end{aligned} \tag{A2}$$

Equations (A2), in the coordinates (x_2, y_2), represent a straight line and an ellipse. The bounds on C_1 and C_2 become more transparent if the ellipse were transformed into a circle. Let new coordinates (α_1, β_1) be defined by the transformation

$$\sqrt{2}x_2 = \frac{\alpha_1}{\sqrt{1+c}} - \frac{\beta_1}{\sqrt{1-c}}$$

$$\sqrt{2}y_2 = \frac{\alpha_1}{\sqrt{1+c}} + \frac{\beta_1}{\sqrt{1-c}}$$

Applying this transformation to Eqs. (A2), we obtain

$$\begin{aligned} \frac{(d_1+d_2)}{\sqrt{1+c}}\alpha_1 + \frac{(d_2-d_1)}{\sqrt{1-c}}\beta_1 &= 2(d_3 + C_2^2 - C_1^2) \\ \alpha_1^2 + \beta_1^2 &= C_2^2 - z_2^2 \end{aligned} \tag{A3}$$

Graphically Eqs. (A3) represent a straight line and a circle. These two curves possibly intersect in two, one, or zero points. The minimum distance from the origin to the straight line is

$$d = \frac{|s(d_3 + C_2^2 - C_1^2)|}{\sqrt{d_1^2 + d_2^2 - 2cd_1d_2}} \tag{A4}$$

where $s^2 = 1 - c^2$. For a real solution, we must have

$$d \le \sqrt{C_2^2 - z_2^2}$$

or

$$(d_3 + C_2^2 - C_1^2)^2 \le d_4^2(C_2^2 - z_2^2) \tag{A5}$$

where

$$d_4^2 = (d_1^2 + d_2^2 - 2cd_1d_2)/s^2$$

The lower bounds of C_1^2 and C_2^2 are obtained more easily by first performing two transformations. The first transformation is a rotation by $\pi/4$ rad defined by the equations

$$\sqrt{2}C_3 = C_1^2 + C_2^2 \qquad \sqrt{2}C_4 = C_2^2 - C_1^2$$

The second transformation is a translation defined by the equations

$$C_5 = C_3 + (\sqrt{2}/16)(d_4^2 - 16z_2^2 - 8d_3)$$

$$C_6 = C_4 + (1/4\sqrt{2})(4d_3 - d_4^2)$$

These two transformations reduce the inequality expressed in Eq. (A5) to the inequality

$$C_6^2 \le \frac{d_4^2}{2\sqrt{2}} C_5 \tag{A6}$$

Equation (A6) represents a parabola in (C_5, C_6) coordinates. In Fig. 5 the two transformation coordinates [(C_3, C_4) rotation, and (C_5, C_6)-translation] and the inequality expressed by Eq. (A6) are illustrated in the original (C_1^2, C_2^2) coordinate system.

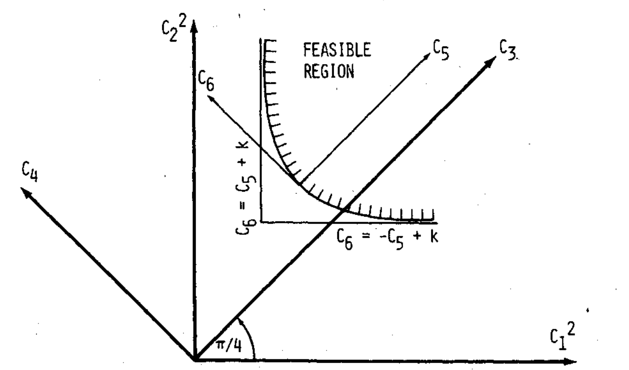

Fig. 5 Feasible region illustrated in the original (C_1^2, C_2^2), rotated (C_3, C_4), and translated (C_5, C_6) coordinate systems.

Lower Bound on C_1^2

The limiting (bounding) value of C_1^2 can be seen from Fig. 5 to be defined by the line

$$C_6 = C_5 + k \tag{A7}$$

where k is the intercept on the C_6 axis. A value for this intercept is obtained by using this straight line in Eq. (A6) to obtain

$$k \leq \frac{\sqrt{2} d_4^2}{16} \tag{A8}$$

Using Eqs. (A7) and (A8) and the two transformations, a lower bound for C_1^2 is determined as

$$C_1^2 \geq d_3 + z_2^2 - (d_4^2/4) \tag{A9}$$

Lower Bound on C_2^2

From Fig. 5 the limiting (bounding) value on C_2^2 is defined by the line

$$C_6 = -C_5 + k \tag{A10}$$

where the intercept for this line must satisfy the inequality

$$k \geq -(\sqrt{2}/16) d_4^2 \tag{A11}$$

Using Eqs. (A10) and (A11) and the two transformations, a lower bound for C_2^2 is determined as

$$C_2^2 \geq z_2^2 \tag{A12}$$

The interpretation of the lower bound on C_2^2 is physically clear. If a solution is to be feasible, there certainly must be a sufficient amount of ΔV_2 to achieve the final orbital plane from the transfer orbit.

The bound on C_1^2 [Eq. (A9)] is complicated by the dual requirements of a sufficient amount of ΔV to 1) perform the plane change from the initial orbit to the transfer orbit, and 2) place the vehicle on a conic that will intercept the final orbit.

Similary, from Eqs. (8), (9), (12), and (14a), another set of lower bounds on C_1^2 and C_2^2 can be determined as

$$C_1^2 \geq z_1^2 \tag{A13}$$

and

$$C_2^2 \geq \tilde{d}_3 + z_1^2 - (\tilde{d}_4^2/4) \tag{A14}$$

where

$$\tilde{d}_3 = 2\tilde{m}/r_2$$

$$\tilde{d}_4^2 = (\tilde{d}_1^2 + \tilde{d}_2^2 - 2c\tilde{d}_1\tilde{d}_2)/s^2$$

$$\tilde{d}_1 = 2(a_1 - a_3)$$

$$\tilde{d}_2 = 2[r_1(a_3 + a_4)/r_2 + a_2 + a_3]$$

and

$$\tilde{m} = r_2(E_f - E_i + z_1^2 - z_2^2 - a_3 b_3) + (a_1 + a_4) b_1$$

The lower bounds on C_1^2 and C_2^2 are selected as the maximum values of Eqs. (A9) and (A13) and Eqs. (A12) and (A14), respectively.

$$\begin{aligned} C_1^2 &= \max[d_3 + z_2^2 - (d_4^2/4), z_1^2] \\ C_2^2 &= \max[z_2^2, \tilde{d}_3 + z_1^2 - (\tilde{d}_4^2/4)] \end{aligned} \tag{A15}$$

Acknowledgment

The assistance of R. Gonzalez, NASA Johnson Space Center, is gratefully acknowledged for his aid in obtaining the graphical data.

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